

THETA CORRESPONDENCE FOR $U(1, 1)$ AND $U(2)$

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ABSTRACT. In this paper, we parametrize certain irreducible supercuspidal representations of $U(1, 1)$ and $U(2)$ via explicit induction data. The parametrization depends on traceless elements of negative valuation in a quadratic extension of base field. We use the lattice model of the Weil representation to determine which traceless elements are involved in the theta correspondence for reductive dual pair $U(1, 1)$ and $U(2)$.

1. INTRODUCTION

Let F be a p -adic field with odd residual characteristic. Let E be a quadratic extension of F . Let D be the quaternion division algebra over F equipped with the nondegenerate symmetric bilinear form defined by the norm map $N_{D/F}$. Let $(V, (\cdot, \cdot)_2)$ be a two-dimensional skew-Hermitian space over E with a hyperbolic pair as basis and let $U(1, 1)$ be the corresponding isometry group. Let $(W, (\cdot, \cdot)'_2)$ be a two-dimensional Hermitian vector space over E with $W = D$ and let $U(2)$ be the corresponding isometry group. Then $(U(1, 1), U(2))$ forms a reductive dual pair in the symplectic group $Sp(\mathcal{W})$ where $\mathcal{W} = V \otimes_E W$ is a nondegenerate symplectic vector space equipped with the tensor product of the forms.

Let χ be a nontrivial additive character of F . Let ω_χ be the corresponding Weil representation associated to the metaplectic cover of $Sp(\mathcal{W})$. By restricting the Weil representation, we obtain a correspondence between some irreducible admissible representations of the metaplectic cover of $U(1, 1)$ and some irreducible admissible representations of the metaplectic cover of $U(2)$. This correspondence is known as the *theta correspondence* or *Howe duality*. It is known that the correspondence is one-to-one by R. Howe [12] and J.-P. Waldspurger [19]. In case of unitary groups, there are very few known examples of explicit theta correspondence, such as (U_1, U_1) by Moen [13] and recent results for small unitary groups by Misaghian [11], Stuffelbeam [18] and Pan [14]. In this paper, we focus on two-dimensional unitary groups $U(1, 1)$ and $U(2)$.

The fundamental problem is to parametrize irreducible admissible supercuspidal representations and use the lattice model of the Weil representation to construct explicit vectors that lead to a description of the local theta correspondence for the dual pair. Our approach follows the parametrization methods first used by Kutzko in [6] and [7] for $GL_2(F)$, then by Manderscheid in [8] for $SL_2(F)$. The advantage of this approach is that explicit construction of supercuspidals yields exact parametrizing data.

The contents of this paper are as follows. In section 2, we set up our notation and investigate groups $G = U(1, 1)$ and $G' = U(2)$. In section 3, we outline the induction

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method that we use to construct relevant representations of $U(1,1)$ and $U(2)$. In section 4, we construct some irreducible admissible supercuspidal representations of $U(1,1)$ with explicit induction data. We associate characters to traceless elements of E and proceed with parametrization methods of Kutzko and Manderscheid. We repeat the process in section 5 to construct some irreducible admissible supercuspidal representations of $U(2)$. In section 6, we outline the lattice model of the Weil representation necessary in order to construct explicit vectors leading to the description of the theta correspondence. In section 7, we determine which traceless elements are involved in the correspondence and show that they belong to corresponding conjugacy classes in GL_2 and the division algebra.

2. NOTATION AND STRUCTURE OF $U(1,1)$ AND $U(2)$

Let F be a nonarchimedean local field of residual characteristic p with p odd. Let \mathcal{O}_F be the ring of integers of F , let P_F be the maximal ideal in \mathcal{O}_F , let $\varpi = \varpi_F$ be uniformizer of F and let k_F be the residue field \mathcal{O}_F/P_F with cardinality q . Let v_F denote the valuation of F and \mathcal{O}_F^\times the group of units of F -space. Let E be a quadratic extension of F . Let $\mathcal{O}_E, P_E, k_E, \varpi_E, v_E$ and \mathcal{O}_E^\times play the corresponding roles with respect to E .

Let $\tau : x \mapsto \bar{x}$ denote the nontrivial Galois automorphism of E over F . Let $N = N_{E/F}$ and $Tr = Tr_{E/F}$ be the usual norm and trace maps associated to the extension E/F . Let E^1 be the group of norm-one elements in E and let E^0 denote the traceless elements of E .

Let D be the nonsplit quaternion division algebra over F equipped with the nondegenerate symmetric bilinear form defined by the (reduced) norm map $N_{D/F}$. Let τ_D denote the involution on D such that $N_{D/F}(x) = x\tau_D(x)$ and $Tr_{D/F}(x) = x + \tau_D(x)$.

Let $a \in D, a \notin F$. The F -algebra $F[a]$ is then a field; since D is an $F[a]$ -vector space, we must have $[F[a] : F] = 2$. Further, there exists a separable quadratic extension E/F such that E admits an F -embedding in D , see [1]. More importantly, any quadratic field extension E/F can be embedded in D . This has a profound effect. If E is a quadratic subfield of D , then we may realize D as the cyclic algebra $(E/F, \tau, a)$ where a is an element of F^\times which is not in the image of the norm map $N_{E/F}$. Notice that τ is precisely the restriction of the involution τ_D to E . In particular, given a generator α for E/F there exists an element δ in D^\times such that $\delta\alpha\delta^{-1} = \tau(\alpha) = -\alpha$ and $\delta^2 = a$. We can take δ to be uniformizer of D and let \mathcal{O}_D, P_D, k_D play the corresponding roles. Then $\{1, \alpha, \delta, \alpha\delta\}$ forms a basis of D over F and $D = E \oplus \delta E$.

If E/F is unramified, we take $\varpi_E = \varpi = \delta^2$ and we choose an element z of F , which is not a square in F , so that $E = F(\alpha)$ with $\alpha^2 = z, \alpha \in \mathcal{O}_E^\times$ and $\alpha^2 \in \mathcal{O}_F^\times$. We also have $|k_E| = q^2$.

We realize $G = U(1,1)$ as the isometry group of a two-dimensional skew-Hermitian space V over E having a hyperbolic pair $\{u, v\}$ as basis. In particular,

$$U(1,1) = \left\{ x \in GL_2(E) : \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = x \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \bar{x}^t \right\}$$

For such x , $\det x \cdot \overline{(\det x)} = 1$. An easy argument reveals that $G = SU(1,1) \rtimes E^1$ where $SU(1,1)$ consists of the determinant-one elements of G . There is a natural identification of $SU(1,1)$ and $SL_2(F)$ and henceforth we consider $G = SL_2(F) \rtimes E^1$.

We have three distinct copies of E^1 in G . The first is the semidirect copy, the second is the natural embedding of E^1 into $SL_2(F)$ (from a natural embedding of E^\times into $GL_2(F)$) via

$$x + y\alpha \mapsto \begin{pmatrix} x & y \\ y\alpha^2 & x \end{pmatrix}.$$

The third copy of E^1 is the copy in the center of G given by $\lambda \mapsto (\lambda, \lambda^2)$.

For $G' = U(2)$, we consider the Hermitian form on a two dimensional E -vector space $W = D$ defined by:

$$(u, v)_2 = \frac{1}{2} \text{Tr}_{D/E}(u\bar{v}), \quad u = u_1 + u_2\delta, \quad v = v_1 + v_2\delta, \quad u_1, u_2, v_1, v_2 \in E.$$

It is easy to verify that the form is nondegenerate and anisotropic. We take G' to be an isometry group of $(W, <, >')$.

$$U(2) = \left\{ x \in GL_2(E) : \begin{pmatrix} 1 & 0 \\ 0 & -\delta^2 \end{pmatrix} = x \begin{pmatrix} 1 & 0 \\ 0 & -\delta^2 \end{pmatrix} \bar{x}^t \right\}.$$

Again, there is an important copy of E^1 in G' , namely $\begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}$, $\lambda \in E^1$. The anisotropic $U(2)$ has a subgroup of $SU(2)$ consisting of determinant-one elements and we can identify it with the group of norm-one elements in D^\times , $D^1 = \{x = a + c\delta : a, c \in E, N_{D/F}(x) = 1\}$. Similar argument as above reveals that $G' = D^1 \rtimes E^1$.

3. IRREDUCIBLE ADMISSIBLE REPRESENTATIONS OF G, G'

To obtain irreducible admissible representations of G and G' , we use the following approach. First, we associate characters of F to traceless elements and extend them to characters of certain open compact subgroups of G and G' . Second, we induce these characters using open compact induction to obtain irreducible admissible supercuspidal representations of G, G' . We make a frequent use of the following facts. By Jacquet [5], any irreducible smooth representation of a reductive p -adic group is admissible. And second, any irreducible representation obtained via compact induction from open compact subgroup is supercuspidal ([2]). Thus, if we exhibit irreducibility of a compactly-induced representation, admissibility and supercuspidality automatically follow. The irreducible admissible representations of G and G' were previously studied by Stufflebeam [18] and Misaghian [11] and hence we omit some of the proofs.

4. IRREDUCIBLE ADMISSIBLE REPRESENTATIONS OF $U(1, 1)$

Recall E^1 is the group of norm-one elements in E^\times and E^0 is the group of the traceless elements in E . For any r , let $P_E^r = \{x \in E : x = a\varpi_E^r, \text{ for some } a \in \mathcal{O}_E\}$ and define P_F^r similarly. For $r \geq 1$, let $E_r^1 = \{\lambda \in E^1 : \lambda - 1 \in P_E^r\}$. We start with some important subgroups of $U(1, 1)$.

Henceforth, we assume E/F is unramified. The subgroup $K_1 = SL_2(\mathcal{O}_F) \rtimes E^1$ is a maximal compact open subgroup of $U(1, 1)$. Let

$$w = \begin{pmatrix} 0 & 1 \\ \varpi & 0 \end{pmatrix}$$

Then the other maximal compact open subgroup is, up to conjugacy, $K_2 = w^{-1}K_1w$. Let $SL_2^r(\mathcal{O}_F) = \{x \in SL_2(\mathcal{O}_F) : x - 1 \equiv 0 \pmod{P_F^r}\}$. Let $K_1^r = E^1 SL_2^r(\mathcal{O}_F) \rtimes E^1$ and $K_2^r = w^{-1}K_1^r w$. These are again open compact subgroups of $U(1, 1)$.

Fix a nontrivial additive character ψ of F with the conductor P_F . Since $E = F(\alpha)$, we can naturally identify $\text{End}_F(E)$ with $M_2(F)$ via the map $a + b\alpha \mapsto \begin{pmatrix} a & b\alpha^2 \\ b & a \end{pmatrix}$, $a, b \in F$. Under this map, a traceless element is identified with a traceless matrix. Denote $M_-(F)$ the traceless matrices in $M_2(F)$. In particular for $\beta = y\alpha \in E^0$, we have a natural injection $E^0 \rightarrow M_-(F)$ via

$$\beta = \begin{pmatrix} 0 & y\alpha^2 \\ y & 0 \end{pmatrix}$$

Let $M_-^r(F)$ denote the traceless matrices whose entries all have valuation $\geq r$. Let $n \geq 1$ and $r = \lfloor \frac{n+2}{2} \rfloor$. The map $x \mapsto x - 1$ gives an isomorphism from $M_-^r(F)/M_-^{n+1}(F) \cong SL_2^r(\mathcal{O}_F)/SL_2^{n+1}(\mathcal{O}_F)$. This gives an isomorphism of character groups

$$M_-^{-n}(F)/M_-^{1-r}(F) \cong (SL_2^r(\mathcal{O}_F)/SL_2^{n+1}(\mathcal{O}_F))^\vee$$

which yields a map $b \mapsto \psi_b$ where $\psi_b(x) = \psi(\text{Tr}(b(x-1)))$, see [1].

Proposition 1. *Let $\beta \in E^0$ and $v_E(\beta) = -n$, $n \geq 1$. For $r = \lfloor \frac{n+2}{2} \rfloor$, we have $\psi_\beta : SL_2^r(\mathcal{O}_F) \rightarrow \mathbb{C}^\times$ is a character trivial on $SL_2^{n+1}(\mathcal{O}_F)$.*

Proof. This follows directly from the duality above. \square

Next, we want ψ_β is invariant under the semidirect action of E^1 . From now on, fix $\beta \in E^0$ and $v_E(\beta) = -n$, $n \geq 1$.

Proposition 2. *Let $x \in SL_2^r(\mathcal{O}_F)$. We have $\psi_\beta(\sigma_\lambda(x)) = \psi_\beta(x)$.*

Proof. See [16], [18]. \square

Let $SL_2(\mathcal{O}_F)$ act on ψ_β by conjugation: $\psi_\beta^g(h) = \psi_\beta(ghg^{-1})$.

Lemma 1. *The kernel of the action of $SL_2(\mathcal{O}_F)$ on ψ_β is $E^1 SL_2^{n-r+1}(\mathcal{O}_F)$.*

Proof. See [16], [18]. \square

For $\beta \in E^0$, let $\Lambda_\beta = \{\phi \in (E^1)^\vee : \phi \equiv \psi_\beta \text{ on } E^1 \cap SL_2^r(\mathcal{O}_F)\}$. If $\phi \in \Lambda_\beta$, then the map $\phi_\beta : E^1 SL_2^r(\mathcal{O}_F) \rightarrow \mathbb{C}^\times$ given by $\phi_\beta(\lambda x) = \phi(\lambda)\psi_\beta(x)$ is a well defined character. Let $\eta \in (E^1)^\vee$ and consider $\phi_{(\beta, \eta)} : K_1^r \rightarrow \mathbb{C}^\times$ given by $\phi_{(\beta, \eta)}(g, \gamma) = \phi_\beta(g)\eta(\gamma)$. This is again well defined character.

Again, we can conjugate these characters by w and obtained well defined characters $\phi_{(\beta, \eta)}^w : K_2^r \rightarrow \mathbb{C}^\times$ where $\phi_{(\beta, \eta)}^w(\omega^{-1}(g, \gamma)\omega) = \phi_{(\beta, \eta)}(g, \gamma)$.

In order to construct irreducible admissible supercuspidal representations of $U(1, 1)$, we have to place further conditions on $v_E(\beta) = -n$ such as n odd or even. If $n = 2m + 1$, then $r = m + 1$ and $n - r + 1 = 2m + 1 - m - 1 + 1 = m + 1 = r$ and hence the stabilizer of ψ_β under the action of $SL_2(\mathcal{O}_F)$ agrees with its domain on the $SL_2^*(\mathcal{O}_F)$ -part. If $n = 2m$, then $r = m + 1$ and $n - r + 1 = m = r - 1$ so that the stabilizer of ψ_β under the action of $SL_2(\mathcal{O}_F)$ is strictly bigger on $SL_2^*(\mathcal{O}_F)$ -part than its group of definition.

4.1. Odd Valuation. In this section, we let $\beta \in E^0$ with $v_E(\beta) = -n = -(2m+1)$, $r = m+1$ and let $G = U(1,1)$. For $\eta \in (E^1)^\times$, consider $\pi_{(\beta, \phi, \eta)} = \text{Ind}(G, K_1^r; \phi_{(\beta, \eta)})$.

Theorem 1. $\pi_{(\beta, \phi, \eta)}$ is an irreducible supercuspidal representation of G with central character $\phi \cdot \eta^2$.

Proof. As mentioned above, we only need to show that $\pi_{(\beta, \phi, \eta)}$ is irreducible. Given a set of double coset representatives $\{x_i\}_{i \in I}$ for $E^1 SL_2^r(\mathcal{O}_F) \backslash SL_2(F) / E^1 SL_2^r(\mathcal{O}_F)$, it is clear that

$$G = \bigcup_{i \in I} K_1^r \backslash (x_i, 1) / K_1^r$$

Since the semidirect action of E^1 may have put some representatives in the same double coset, we will take an index subset $S \subset I$ such that S has one representative for each double coset. Then by Mackey theory,

$$I(\pi_{(\beta, \phi, \eta)}, \pi_{(\beta, \phi, \eta)}) \cong \bigoplus_{i \in S} I(\phi_{(\beta, \eta)}, \phi_{(\beta, \eta)}^{x_i})$$

Upon the restriction to $E^1 SL_2^r(\mathcal{O}_F) \cap x_i^{-1} (E^1 SL_2^r(\mathcal{O}_F)) x_i$, $\phi_{(\beta, \eta)} = \phi_\beta$ and $\phi_{(\beta, \eta)}^{x_i} = \phi_\beta^{x_i}$. We also know by [9], that the intertwining of x_i outside the compact subgroup is 0. Thus for each i ,

$$I(\phi_{(\beta, \eta)}, \phi_{(\beta, \eta)}^{x_i}) \subset I(\phi_\beta, \phi_\beta^{x_i})$$

Since n is odd, the stabilizer of ψ_β is $E^1 SL_2^{n-r+1}(\mathcal{O}_F) = E^1 SL_2^r(\mathcal{O}_F)$. Hence by [8], the representation $\text{Ind}(SL_2(F), E^1 SL_2^r(\mathcal{O}_F); \phi_\beta)$ is an irreducible supercuspidal of $SL_2(F)$. Then Mackey theory and the selection of representatives give $I(\phi_\beta, \phi_\beta^{x_i}) = 0$ for all $x_i \neq 1$. Since $I(\phi_{(\beta, \eta)}, \phi_{(\beta, \eta)}) = \mathbb{C}$, we conclude that $I(\pi_{(\beta, \phi, \eta)}, \pi_{(\beta, \phi, \eta)}) = \mathbb{C}$ and then [2], Theorem 11.4 gives us $\pi_{(\beta, \phi, \eta)}$ is an irreducible representation. It follows it is admissible and supercuspidal. The statement about the central character follows from the definition of $\pi_{(\beta, \phi, \eta)}$. \square

With the same notation as above, consider the character $\phi_{(\beta, \eta)}^w$ on K_2^r . Let $\pi'_{(\beta, \phi, \eta)} = \text{Ind}(G, K_2^r; \phi_{(\beta, \eta)}^w)$.

Theorem 2. $\pi'_{(\beta, \phi, \eta)}$ is an irreducible supercuspidal representation of G with the central character $\phi \cdot \eta^2$.

Proof. The construction is virtually the same, for details see [16], [18]. \square

4.2. Heisenberg Construction. In this section, we have $\beta \in E^0$ with $v_E(\beta) = -n = -2m$, $r = m+1$ and again $G = U(1,1)$. Since $n - r + 1 = m = r - 1$, the character ψ_β on $SL_2^r(\mathcal{O}_F)$ is stabilized by subgroup $E^1 SL_2^{r-1}(\mathcal{O}_F)$. Hence for any $\eta \in (E^1)^\times$, $I(G, K_1^r; \phi_{(\beta, \eta)})$ will be reducible and we have to use different methods to find irreducible supercuspidal representations of G . The construction has been studied in [18], therefore we only state the results.

Let $SL_2^r(\mathcal{O}_F)$ be the subset of $SL_2(\mathcal{O}_F)$ such that the diagonal elements are congruent to 1 modulo P_F^{r-1} and off-diagonal elements are congruent to 0 mod P_F^r . Then it is clear that $SL_2^r(\mathcal{O}_F) \subset SL_2^r(\mathcal{O}_F) \subset SL_2^{r-1}(\mathcal{O}_F)$. Also, the character ψ_β can be extended onto $SL_2^r(\mathcal{O}_F)$ since $\beta \in E^0$ and hence ψ_β depends only on off-diagonal elements.

Let $E_0^1 = E^1 \cap F^\times(1 + P_E)$. Then one checks that E_0^1 normalizes $SL_2^r(\mathcal{O}_F)$. Depending on α , E^1 may not normalize the above. Select $\phi \in (E^1)^\wedge$ such that $\phi = \psi_\beta$ on $E^1 \cap SL_2^r(\mathcal{O}_F)$. Define $\phi_\beta : E_0^1 SL_2^r(\mathcal{O}_F) \rightarrow \mathbb{C}^\times$ naturally. It is clear that ϕ_β is a character. Now, when we add our semidirect product action, we may not necessarily obtain a group. Therefore instead of having E^1 in a semidirect product, we will work with $E_1^1 = \{\lambda \in E^1 : \lambda - 1 \in P_E^1\}$. This will guarantee us that $E_0^1 SL_2^r(\mathcal{O}_F) \rtimes E_1^1$ is a subgroup.

The extended ψ_β is invariant under the semidirect action. And the kernel of the $SL_2(\mathcal{O}_F)$ -conjugate action on this extended ψ_β is $E_0^1 SL_2^r(\mathcal{O}_F)$. Let $\eta \in (E^1)^\wedge$ and consider the character $\phi_{(\beta, \eta)} : E_0^1 SL_2^r(\mathcal{O}_F) \rtimes E_1^1 \rightarrow \mathbb{C}^\times$ given by $\phi_{(\beta, \eta)}(\lambda g, \gamma) = \phi_\beta(\lambda g) \eta(\gamma)$ where η is restricted to E_1^1 . For later computations, we need exact number of matrices that form a complete set of distinct coset representatives for various cosets.

Lemma 2. $|E_0^1 SL_2^{r-1}(\mathcal{O}_F) \rtimes E_1^1 : E_0^1 SL_2^r(\mathcal{O}_F) \rtimes E_1^1| = q;$

$$|E_0^1 SL_2^{r-1}(\mathcal{O}_F) \rtimes E_1^1 : E_0^1 SL_2^r(\mathcal{O}_F) \rtimes E_1^1| = q^2;$$

$$|E_0^1 SL_2^r(\mathcal{O}_F) \rtimes E_1^1 \setminus E_0^1 SL_2^{r-1}(\mathcal{O}_F) \rtimes E_1^1 / E_0^1 SL_2^r(\mathcal{O}_F) \rtimes E_1^1| = q;$$

$$|E_0^1 SL_2^r(\mathcal{O}_F) \rtimes E_1^1 \setminus E^1 SL_2^{r-1}(\mathcal{O}_F) \rtimes E_1^1 / E_0^1 SL_2^r(\mathcal{O}_F) \rtimes E_1^1| = \frac{q^2(q+1)}{2};$$

$$|E^1 SL_2^r(\mathcal{O}_F) \rtimes E_1^1 \setminus E^1 SL_2^{r-1}(\mathcal{O}_F) \rtimes E_1^1 / E_0^1 SL_2^r(\mathcal{O}_F) \rtimes E_1^1| = q^2;$$

$$|E^1 SL_2^r(\mathcal{O}_F) \rtimes E_1^1 \setminus E^1 SL_2^{r-1}(\mathcal{O}_F) \rtimes E_1^1 / E^1 SL_2^r(\mathcal{O}_F) \rtimes E_1^1| = 2q - 1$$

Proof. See [16]. □

Consider $\rho_{(\beta, \phi, \eta)}^o = \text{Ind}(E_0^1 SL_2^{r-1}(\mathcal{O}_F) \rtimes E_1^1, E_0^1 SL_2^r(\mathcal{O}_F) \rtimes E_1^1; \phi_{(\beta, \eta)})$. This is a q -dimensional irreducible representation.

Consider $\rho_{(\beta, \phi, \eta)} = \text{Ind}(E_0^1 SL_2^{r-1}(\mathcal{O}_F) \rtimes E_1^1, E_0^1 SL_2^r(\mathcal{O}_F) \rtimes E_1^1; \phi_{(\beta, \eta)})$. This representation decomposes with the respect to $\rho_{(\beta, \phi, \eta)}^o$. By Lemma 2, we have exactly q copies.

$$\text{Let } \tau_{(\beta, \phi, \eta)} = \text{Ind}(E^1 SL_2^{r-1}(\mathcal{O}_F) \rtimes E_1^1, E^1 SL_2^r(\mathcal{O}_F) \rtimes E_1^1; \phi_{(\beta, \eta)}).$$

Proposition 3. *Let f be the character of $\tau_{(\beta, \phi, \eta)}$ and g the character of $\text{Ind}(E^1 SL_2^{r-1}(\mathcal{O}_F) \rtimes E_1^1, E_0^1 SL_2^r(\mathcal{O}_F) \rtimes E_1^1; \phi_{(\beta, \eta)})$. Then $2q^{-1}g - f$ is the character of a q -dimensional irreducible representation $\tau_{(\beta, \phi, \eta)}^1$ of $E^1 SL_2^{r-1}(\mathcal{O}_F) \rtimes E_1^1$ whose restriction to $E_0^1 SL_2^{r-1}(\mathcal{O}_F) \rtimes E_1^1$ is $\rho_{(\beta, \phi, \eta)}^o$.*

Proof. For details, see [16], [18]. □

Lemma 3. *The representation $\tau_{(\beta, \phi, \eta)}^1$ extends to a unique q -dimensional irreducible representation of K_1^{r-1} .*

Proof. For detailed construction, see [18]. □

$$\text{Define } \pi_{(\beta, \phi, \eta)} = \text{Ind}(G, K_1^{r-1}; \tau_{(\beta, \phi, \eta)}^1).$$

Theorem 3. $\pi_{(\beta, \phi, \eta)}$ is an irreducible supercuspidal representation of G with the central character $\phi \cdot \eta^2$.

Proof. The argument is analogous to that involved in proving Theorem 4.1.1. □

Keeping the same β, η, r, ϕ, n , we will construct irreducible supercuspidals on K_2^r . Consider $\phi_{(\beta, \eta)}^w$ on K_2^r . We can use analogous arguments, properly modify them and reproduce

the unique q -dimensional irreducible representation τ^1 of K_2^{r-1} . Hence we only list the following result. Let $\pi'_{(\beta, \phi, \eta)} = \text{Ind}(G, K_2^{r-1}; \tau^1_{(\beta, \phi, \eta)})$.

Theorem 4. $\pi'_{(\beta, \phi, \eta)}$ is an irreducible supercuspidal representation of G with the central character $\phi \cdot \eta^2$.

4.3. The Level Zero Case. In [4], Gerardin defines the Weil representation for symplectic groups, general linear groups and unitary groups over finite fields. He canonically identifies $U(2, k_E) \cong SL_2(k_F) \rtimes k_E^1$ with a subgroup of $Sp(4, k_F)$. The results are applicable only if E/F is unramified.

Let χ be an additive character of k_F and ω_χ be the associated Weil representation of $Sp(4, k_F)$. Gerardin proves that the Weil representation restricted to $U(2, k_E)$ decomposes into irreducibles,

$$\omega_{\chi|U(2, k_E)} = \text{sgn} \otimes \left(\bigoplus_{\xi \in (E^1/E_1^1)^\vee} \vartheta_\xi \right)$$

where sgn is the unique nontrivial quadratic character of $U(2, k_E)$ and ϑ_1 is a q -dimensional irreducible representation with the central character 1, and for $\xi \neq 1$, ϑ_ξ is a $(q-1)$ -dimensional irreducible cuspidal representation of $U(2, k_E)$ with the central character ξ .

We assume $\xi \neq 1 \in (E^1/E_1^1)^\vee$ and use the corresponding cuspidal representation ϑ_ξ to construct irreducible admissible supercuspidals of $U(1, 1)$. We may lift ϑ_ξ to an irreducible $(q-1)$ -dimensional representation $\rho_{(\xi, \eta)}$ of K_1 . The construction and results are known, hence we only state the important theorems.

Define $\pi_{(\xi, \eta)} = \text{Ind}(G, K_1; \rho_{(\xi, \eta)})$.

Theorem 5. $\pi_{(\xi, \eta)}$ is an irreducible admissible supercuspidal representation of G with central character $\xi \cdot \eta^2$.

Proof. See [16, 18]. □

In similar manner, we construct supercuspidals from K_2 . For a nontrivial character $\xi \in (E^1/E_1^1)^\vee$ and $\eta \in (E^1)^\vee$, the representation $\rho_{(\xi, \eta)}^w$ is irreducible on K_2 . Define $\pi'_{(\xi, \eta)} = \text{Ind}(G, K_2; \rho_{(\xi, \eta)}^w)$. The induced representation is an irreducible supercuspidal of G with central character $\xi \cdot \eta^2$.

5. IRREDUCIBLE ADMISSIBLE REPRESENTATIONS OF $U(2)$

Recall $G' = U(2) = D^1 \rtimes E^1$ where D^1 are norm-one elements of D . Also, recall that δ is uniformizer of D with $\delta^2 = \varpi$. For any r , let $P_D^r = \{x \in D : x = a\delta^r, a \in \mathcal{O}_D\}$. Let D^0 denote the traceless elements in D and let $D_r^1 = \{x \in D^1 : x - 1 \in P_D^r\}$. Also notice that, due to ramification of D over F , we have $F \cap P_D^n \subset P_F^{\lfloor \frac{n+1}{2} \rfloor}$. Since G' is compact, all of its irreducible admissible representations are supercuspidal. The supercuspidal representation of $U(2)$ were previously studied in [11]. Our approach provides the explicit construction of supercuspidals with exact parametrizing date. The construction is virtually the same as for case $U(1, 1)$ and hence we only state the important results.

5.1. Characters of D^1 . First, we look at one dimensional irreducible representations. Let $D_1^1 = \{x \in D^1 : x - 1 \in P_D\}$. It is a well known fact that the commutator group of G' , $[G', G'] = D_1^1 \rtimes \{1_E\}$. Straightforward computations show that D^1/D_1^1 is a cyclic group of order $q + 1$.

Lemma 4. *There is a bijection between characters of D^1 and characters of D^1/D_1^1 .*

Proof. Clear. \square

5.2. Characters Associated to Traceless Elements. This construction is virtually the same as in the case of $U(1, 1)$. Fix a nontrivial additive character ψ of F with the conductor P_F . Recall, E/F is unramified.

Proposition 4. *Let $\beta \in D^0$ and $v_D(\beta) = -n$, $n \geq 1$. For $r = \lfloor \frac{n+2}{2} \rfloor$, define $\psi_\beta : D_r^1 \rightarrow \mathbb{C}^\times$ by $\psi_\beta(h) = \psi(\text{Tr}(\beta(h - 1)))$, $h \in D_r^1$. Then ψ_β is a character of D_r^1 trivial on D_{n+1}^1 .*

Next we want to show ψ_β is invariant under the semidirect action of E^1 . From now on fix $\beta \in D^0$ and $v_D(\beta) = -n$, $n \geq 1$.

Proposition 5. *Let $h \in D_r^1$. We have $\psi_\beta(\sigma'_\lambda(h)) = \psi_\beta(h)$, for $\lambda \in E^1$.*

Proof. Modify the proof of Proposition 2, Section 4. For details, see [16]. \square

For $g \in D^1$, define $\psi_\beta^g(h) = \psi_\beta(ghg^{-1})$. The action is well defined, since for chosen g , $ghg^{-1} \in D_r^1$ for $x \in D_r^1$. Thus we can determine the stabilizer.

Lemma 5. *The stabilizer of the action of D^1 on ψ_β is $E^1 D_{n-r+1}^1$.*

Proof. Simiarly to the proof of Lemma 1, Section 4. \square

For $\beta \in D^0$, let $\Lambda_\beta = \{\gamma \in (E^1)^\wedge : \gamma \equiv \psi_\beta \text{ on } E^1 \cap D_r^1\}$. If $\gamma \in \Lambda_\beta$, then the map $\gamma_\beta : E^1 D_r^1 \rightarrow \mathbb{C}^\times$ given by $\gamma_\beta(\lambda x) = \gamma(\lambda) \psi_\beta(x)$ is a well defined character. Let $\zeta \in (E^1)^\wedge$ and consider $\gamma_{(\beta, \zeta)} : E^1 D_r^1 \rtimes E^1 \rightarrow \mathbb{C}^\times$ given by $\gamma_{(\beta, \zeta)}(g, \lambda') = \gamma_\beta(g) \zeta(\lambda')$. This is again well defined character.

In order to construct the irreducible admissible supercuspidal representations of $U(2)$, we have to place further conditions on $v_D(\beta) = -n$, such as n odd or even. If $n = 2m + 1$, then $r = m + 1$ and $n - r + 1 = 2m + 1 - m - 1 + 1 = m + 1 = r$ and hence the stabilizer of φ_β under the action of D^1 and its domain agrees on D_*^1 -part. If $n = 2m$, then $r = m + 1$ and $n - r + 1 = m = r - 1$ so that the stabilizer of ψ_β under the action of D^1 could be bigger on the D_*^1 -part than its group of definition. Thus we will have to place additional conditions on r , such as r is odd or even.

5.3. Odd Valuation. Let $\beta \in D^0$ with $v_D(\beta) = -(2m + 1)$, $r = m + 1$, and let $G' = U(2)$. For $\zeta \in (E^1)^\wedge$, consider $\pi_{(\beta, \gamma, \zeta)} = \text{Ind}(G', E^1 D_r^1 \rtimes E^1; \gamma_{(\beta, \zeta)})$.

Theorem 6. *$\pi_{(\beta, \gamma, \zeta)}$ is an irreducible admissible representation of G' .*

Proof. Given the set of double coset representatives $\{x_i\}_{i \in I}$ for $E^1 D_r^1 \backslash D^1 / E^1 D_r^1$, it is clear that

$$G' = \bigcup_{i \in I} E^1 D_r^1 \rtimes E^1 \backslash (x_i, 1) / E^1 D_r^1 \rtimes E^1$$

Since the semidirect action of E^1 may have related some representatives, we will take an index subset $J \subset I$ such that J has one representative for each double coset. Then by Mackey Theory,

$$I(\pi_{(\beta, \gamma, \zeta)}, \pi_{(\beta, \gamma, \zeta)}) \cong \bigoplus_{i \in J} I(\gamma_{(\beta, \zeta)}, \gamma_{(\beta, \zeta)}^{x_i})$$

Upon the restriction to $E^1 D_r^1 \cap x_i^{-1}(E^1 D_r^1)x_i$, $\gamma_{(\beta, \zeta)} = \gamma_\beta$ and $\gamma_{(\beta, \zeta)}^{x_i} = \gamma_\beta^{x_i}$. Thus for each i ,

$$I(\gamma_{(\beta, \zeta)}, \gamma_{(\beta, \zeta)}^{x_i}) \subset I(\gamma_\beta, \gamma_\beta^{x_i})$$

Since n is odd, the stabilizer of ψ_β is exactly the domain of it and hence by Clifford Theory, theorem (45.2)' in [3], the representation $\text{Ind}(D^1, E^1 D_r^1; \gamma_\beta)$ is an irreducible supercuspidal of D^1 . Then Mackey theory and the selection of representatives give $I(\gamma_\beta, \gamma_\beta^{x_i}) = 0$ for $x_i \neq 1$. Since $I(\gamma_{(\beta, \zeta)}, \gamma_{(\beta, \zeta)}) = \mathbb{C}$, we conclude that $I(\pi_{(\beta, \gamma, \zeta)}, \pi_{(\beta, \gamma, \zeta)}) = \mathbb{C}$. By [1], Theorem 11.4 it follows that $\pi_{(\beta, \gamma, \zeta)}$ is irreducible admissible representation and since $U(2)$ compact, it is supercuspidal. \square

5.4. Even Valuation. In this section, we take $v_D(\beta) = -n = -(2m)$, then $r = m + 1$ and $n - r + 1 = m = r - 1$ so that the stabilizer of ψ_β under the action of D^1 could be bigger than its domain. In order to find irreducible admissible representations, we have to place additional conditions on r , such as r is odd or even. For this section, we assume r is odd. It follows m is even and n is divisible by 4. Let D^* denote the set of elements x in D such that $1 + x$ is invertible. Then recall that the Cayley transform is the well defined map c from D^* to itself defined by $c(x) = (1 - x)(1 + x)^{-1}$. Notice that c is a bijection onto D^* with inverse c itself.

Lemma 6. *If r odd, then*

$$(E^1 D_{r-1}^1)/D_{n+1}^1 = (E^1 D_r^1)/D_{n+1}^1$$

where $r - 1 = m = n/2$ and $r = m + 1 = n/2 + 1$.

Proof. Since E/F is unramified, $\mathbf{k}_E = \mathbf{k}_F$. Let $h = \frac{(1 - a\delta^{r-1})}{(1 + a\delta^{r-1})} D_{n+1}^1$ be an element of D_{r-1}^1/D_{n+1}^1 . Notice that we can write h in this form due to the Cayley transform. Write $a = a_0 + a_1\delta$ where $a_0, a_1 \in \mathcal{O}_E$. Now we have:

$$\begin{aligned} h &= \frac{1 - a\delta^{r-1}}{1 + a\delta^{r-1}} D_{n+1}^1 = \frac{1 - (a_0 + a_1\delta)\delta^{r-1}}{1 + (a_0 + a_1\delta)\delta^{r-1}} D_{n+1}^1 \\ &= \left(\frac{1 - a_0\delta^{r-1}}{1 + a_0\delta^{r-1}} \right) \left(\frac{1 - a_1\delta^r}{1 + a_1\delta^r} \right) \left(\frac{1 - (1 - a_0\delta^{r-1})^{-1}(1 - a_1\delta^r)^{-1}(a_1 a_0 \delta^{n+1})}{1 - (1 + a_0\delta^{r-1})^{-1}(1 + a_1\delta^r)^{-1}(a_1 a_0 \delta^{n+1})} \right) \end{aligned}$$

Since r odd, $r - 1 = m$ even and hence $a_0\delta^{r-1} \in E$ and thus the first quotient is in E^1 . By definition, the second quotient is in $D_r^1 = D_{n/2+1}^1$. To obtain the result, it suffices to show that third quotient is in D_{n+1}^1 . Quick calculations shows the quotient is in D^1 and after subtracting 1, the quotient is $\equiv 0 \pmod{P_D^{n+1}}$. Thus

$$(E^1 D_{r-1}^1)/D_{n+1}^1 \subset (E^1 D_r^1)/D_{n+1}^1$$

The other containment is true by definition of the filtration, and we are done. \square

Thus any character of $(E^1 D_{r-1}^1)/D_{n+1}^1$ is a character of $(E^1 D_r^1)/D_{n+1}^1$ and vice versa. Hence we can take ψ_β as in Proposition 1, and obtain a character $\gamma_{(\beta, \zeta)}$. By above Lemma, they both are characters on $E^1 D_{r-1}^1 \rtimes E^1$ and hence the stabilizer of ψ_β under the action of D^1 coincides with its domain. Consider $\pi'_{(\beta, \gamma, \zeta)} = \text{Ind}(G', E^1 D_{r-1}^1 \rtimes E^1; \gamma_{(\beta, \zeta)})$.

Theorem 7. $\pi'_{(\beta, \gamma, \zeta)}$ is an irreducible admissible representation of G' .

Proof. Apply the proof of Theorem 4.3.1 with $r = r - 1$. \square

5.5. Even Valuation - Heisenberg Construction. In this case, we have n -even, r -even. Hence the stabilizer of ψ_β under the action of D^1 is $E^1 D_{r-1}^1$, strictly bigger than its domain. The Heisenberg construction is the same as in case $U(1, 1)$, hence we only state important results.

Lemma 7. There is a unique q -dimensional irreducible representation $\tau_{(\beta, \gamma, \zeta)}^1$ of $E^1 D_{r-1}^1 \rtimes E^1$.

Proof. For details, see [16]. \square

Define $\pi'_{(\beta, \gamma, \zeta)} = \text{Ind}(G', E^1 D_{r-1}^1 \rtimes E^1; \tau_{(\beta, \gamma, \zeta)}^1)$.

Theorem 8. $\pi'_{(\beta, \gamma, \zeta)}$ is an irreducible representation of G' .

Proof. Simiarly to the proof of Theorem 3, Section 4.2. \square

6. LATTICE MODEL OF THE WEIL REPRESENTATION

In this section, we detail the method of lattice models of the Weil representation. This will allow is to explicitly determine the occurence of irreducible admissible representations of $U(1, 1)$ and $U(2)$ as quotients of the smooth Weil representation. This section is a recapitulation of a material in [14] and [8] modified to fit our needs in the next section.

Let $(\mathcal{W}, <, >)$ be a nondegenerate symplectic vector space of dimension $2n$ over F and let $H(\mathcal{W})$ be the associated Heisenberg group, $H(\mathcal{W}) = \mathcal{W} \oplus F$ with F being the center of $H(\mathcal{W})$. Let χ be a notrivial additive character of F and let ρ_χ the associated unique unitary representation of $H(\mathcal{W})$ with central character χ . Let ω_χ be the correspoding Weil representation and denote $\omega_\chi^\infty, \rho_\chi^\infty$ corresponding smooth representations.

In this section, we obtain a realization of ω_χ by working with certain non-self-dual lattices in \mathcal{W} . We begin recalling some features of the Weil representation over the finite field k_F .

Let \mathcal{W}' be a finite dimensional vector space over k . Suppose that \mathcal{W}' is equipped with a nondegenerate skew-symmetric bilinear form $<, >$ and let $G(\mathcal{W}')$ be its isometric group. Let $H(\mathcal{W}')$ denote the Heisenberg group attached to \mathcal{W}' and let χ be a nontrivial additive character of k . Then, there is a unique (up to equivalence) unitary representation of ρ_χ of $H(\mathcal{W}')$ with central character χ . Moreover, there is a representation ω_χ of G on the space ρ_χ such that

$$\omega_\chi(g)\rho_\chi(h) = \rho_\chi(gh)\omega_\chi(g)$$

where $h \in H(\mathcal{W}'), g \in G$. This representation is unique up to equivalence except in the case where $\dim_F \mathcal{W}' = 2$ and $|k| = 3$. In this exceptional case, we may fix ω_χ to satisfy a certain condition in a Schrodinger model. For more details, see [8].

We now turn to the lattice model for a certain type of lattice that is not self-dual. We return to the notation of the previous section and suppose L is an \mathcal{O}_F -lattice in \mathcal{W}' which although is not self-dual does satisfy

$$P_F L^* \subseteq L \subsetneq L^*$$

We will call this lattice a *good lattice*. Notice that $\bar{L} = L^*/L$ is an even dimensional vector space over k .

Let d be an interger such that χ is trivial on P_F^d but not trivial on P_F^{d-1} . Let \bar{x} and \bar{y} in \bar{L} be preimages of x and y in L^* and set $\langle \bar{x}, \bar{y} \rangle_d = \varpi^{1-d} \langle x, y \rangle$. One can check that \langle, \rangle_d is well-defined nondegenerate skew-symmetric bilinear form on \bar{L} . We may also define a character χ' of k by setting $\chi'(\bar{x}) = \chi(x)$ where $\bar{x} \in k$ and x is an element of P_F^{d-1}/P_F^d with image \bar{x} under the map induced by $y \mapsto \varpi^{1-d}y$ from P_F^{d-1} to \mathcal{O}_F . Let $\rho_{\chi'}$ denote a representation of $H(\bar{L})$ with central character χ' .

Let J^* be the subgroup of $H(\mathcal{W}')$ generated by $e(L^*)$ and let J be the subgroup of $H(\mathcal{W}')$ generated by $e(L)$. Then we may inflate $\rho_{\chi'}$ to a representation of J^* which is trivial on J . We also define ρ_L a representation of $\gamma^{-1}(L^*)$ on the space of $\rho_{\chi'}$ by $\rho_L(ah)v = \chi(a)\rho_{\chi'}(h)v$ where $a \in Z(H(\mathcal{W}')), h \in J^*, v$ in the space of $\rho_{\chi'}$. Then $\text{Ind}(H(\mathcal{W}'), \gamma^{-1}(L^*); \rho_L)$ realizes ρ_χ . For more details and proofs, see [8].

We now need to make this realization more explicit. Let X be the finite dimensional Hilbert space of ρ_L and let $\| \cdot \|$ denote the norm on X . Let Y denote the space of ρ_χ and let S_L denote the set of coset representatives for \mathcal{W}'/L^* . Then Y is the set of functions $f : \mathcal{W}' \rightarrow X$ satisfying:

- (i) $f(w+a) = \chi(\langle w, a \rangle / 2) \rho_L(e(a))f(w)$ for $a \in L^*$
- (ii) $\sum_{w \in S_L} \|f(w)\|^2 < \infty$

The action of ρ_χ is given by

$$(\rho_\chi(e(w))f)(w') = \chi(\langle w', w \rangle / 2) f(w' + w)$$

for $f \in Y, w, w' \in \mathcal{W}'$. For each $w \in \mathcal{W}', x \in X$ of length one, let $y_{w,x}$ denote the function on Y supported on $-w + L^*$ taking the value x at $-w$. Then, if we choose an orthonormal basis S_X for X , we have that Y consists of linear combinations

$$\sum_{\substack{w \in S_L \\ w \in S_X}} a_{w,x} y_{w,x}$$

with

$$\sum |a_{w,x}|^2 < \infty$$

and Y^∞ is the subspace of Y consisting of finite linear combinations of the above form.

We now consider ω_χ . Let K be the maximal compact subgroup of $Sp(\mathcal{W}')$ which stabilizes L^* and let K' be a subgroup of K acting trivially on L^*/L . We may identify K/K' with the isometry group of the symplectic space \bar{L} and thus there exists a unique representation ω_χ of K on X which is trivial on K' and satisfies

$$\omega_L(g)\rho_L(h) = \rho_L(gh)\omega_L(g)$$

for h in $\gamma^{-1}(L^*)$ and $g \in K$.

Proposition 6. *The representation ω_χ may be chosen so that it restricts to a representation of K . In particular, ω_χ may be chosen so that for f in Y and k in K*

$$\omega_\chi(k)f(w) = \omega_L(k)f(k^{-1}w)$$

and thus for k in K

$$\omega_\chi(k)y_{w,x} = y_{kw, \omega_L(k)x}.$$

In addition, the space of smooth vectors Y^∞ for ω_χ consists of those f in Y supported on a finite number of W'/L^* cosets, i.e. those f which are finite linear combinations of the $\{y_{w,x}\}$.

Proof. See, for example, Chapter 5 of [12]. \square

Now suppose, L is a lattice in W' as above and M is a sublattice of L . Then $H_M = \{g \in G | (g-1)M^* \subset L^*\}$ is a subgroup of G . And further, we have

Proposition 7. *If a function f in Y is supported on M^* , then*

$$\omega_\chi(h)f(w) = \rho_L(2c(h)w)\chi(<w, c(h)w>)f(w)$$

for h in H_M where $c(h) = (1-h)(1+h)^{-1}$ is the Cayley transform of h .

Proof. This result can be proved with a straightforward modification of the proof of [12]. \square

7. THETA CORRESPONDENCE

In this section, we use the methods of previous section to begin to determine which irreducible admissible representations of $U(1,1)$ and $U(2)$ occur in the theta correspondence. For additive character of ψ fixed in section 4, 5, set $\chi = \psi_\varpi$, that is, $\chi(x) = \psi_\varpi(x) = \psi(\varpi x)$ for x in F . Then $\psi \cdot \text{Tr}_{E/F}$ is a character of E with conductor P_E . Recall $\mathcal{W} = V \otimes_E W$ is equipped with a nondegenerate skew-symmetric bilinear form $\langle\langle, \rangle\rangle$ by setting $\langle\langle v_1 \otimes w_1, v_2 \otimes w_2 \rangle\rangle = \text{Tr}(\langle v_1, v_2 \rangle \langle w_1, w_2 \rangle')$.

Also recall $G = U(1,1)$ is the isometry group of \langle, \rangle_2 , $G' = U(2)$ is the isometry group of \langle, \rangle'_2 . We may identify G and G' with subgroups of $Sp(8)$ by letting G act on \mathcal{W} by premultiplication by inverses and letting G' act on \mathcal{W} by postmultiplication. Note that in this identification G and G' are each other commutants in $Sp(8)$, i.e. form a reductive dual pair. Recall E/F is unramified.

Let $\Gamma = \mathcal{O}_E u + \mathcal{O}_E v$ be the lattice in V and $\Gamma' = \mathcal{O}_E + \mathcal{O}_E \delta$ be the lattice in W . Then $A = \Gamma \otimes \Gamma'$ is a lattice in \mathcal{W} .

Lemma 8. *A is a non-self dual lattice and $A^* = (\mathcal{O}_E u + \mathcal{O}_E v) \otimes (\mathcal{O}_E + P_E^{-1} \delta)$.*

Proof. This can be checked directly, see for example [14], [16]. \square

Notice that A is a non-self dual “good” lattice, i.e. satisfying

$$\varpi_F A^* \subseteq A \subset A^*$$

and hence we may apply results detailed in previous section. Let $A_F(\mathcal{W}) = \text{Hom}_F(\mathcal{W}, \mathcal{W})$ and for $k \in \mathbb{Z}$, $M^k = P^k A$. Let $M = \{M^k\}_{k \in \mathbb{Z}}$ be a lattice chain and let \mathcal{A} be the subring of $A_F(\mathcal{W})$ consisting of elements x such that $xM^k \subseteq M^k$ for all k . Also, for $n \geq 1$, let \mathcal{P}^n be the set of elements x in \mathcal{A} satisfying $xM^k \subseteq M^{k+n}$ for all k . Let $U(\mathcal{A}) = \{x \in Sp(\mathcal{W}) : x \in \mathcal{A}^\times\}$

and for $n \geq 1$, $U^n(\mathcal{A}) = \{x \in Sp(\mathcal{W}) : x - 1 \in \mathcal{P}^n\}$. Finally, let $U_1^n(\mathcal{A}) = U^n(\mathcal{A}) \cap G$ and $U_2^n(\mathcal{A}) = U^n(\mathcal{A}) \cap G'$. Notice that these filtrations correspond to the filtrations on G and G' defined in Sections 4 and 5.

Lemma 9. *For $k \geq 0$, $(M^k)^* = P^{-k}A^*$.*

Proof. Recall $M^k = P^kA$, hence $M^k = (P^k\Gamma) \otimes \Gamma'$ or $M^k = \Gamma \otimes (P^k\Gamma')$. In the first case one can check, $(M^k)^* = (P^{-k}\Gamma) \otimes (\Gamma')^* = P^{-k}A^*$. In the latter case, $(M^k)^* = \Gamma \otimes (P^{-k})(\Gamma')^* = P^{-k}(\Gamma \otimes (\Gamma')^*) = P^{-k}A^*$. \square

Now to use a result of Section 5.2, we fix a set S_A of coset representatives for \mathcal{W}/A^* . Recall for $v \in V, w \in W, x \in X$, $y_{v \otimes w, x}$ denotes the function f in Y supported on $-(v \otimes w) + A^*$ and taking the value x at $-(v \otimes w)$. Also recall $\bar{A} = A^*/A$ is a 4-dimensional vector space over k_F .

Lemma 10. *Let $v, v' \in V, w, w' \in W$ and $x, x' \in X$. Then $y_{v' \otimes w', x'} = cy_{v \otimes w, x}$ for some $c \in \mathbb{C}^\times$ if and only if $v' \otimes w' - v \otimes w \in A^*$ and $x' = bx$ for some $b \in \mathbb{C}$.*

Proof. If $y_{v' \otimes w', x'} = cy_{v \otimes w, x}$ for some $c \in \mathbb{C}^\times$, the supports of the two functions are identical. Thus, $-(v \otimes w) + A^* = -(v' \otimes w') + A^*$ so that $v \otimes w - v' \otimes w' \in A^*$. Also, we have $y_{v' \otimes w', x'}(v' \otimes w') = cy_{v \otimes w, x}(v \otimes w)$ which means

$$\begin{aligned} \chi(\langle\langle v' \otimes w', v' \otimes w' \rangle\rangle / 2)x' &= c\chi(\langle\langle v \otimes w, v \otimes w \rangle\rangle / 2)x \\ x' &= c\chi(\langle\langle v \otimes w, v' \otimes w' \rangle\rangle / 2)x \end{aligned}$$

Conversely, assume $v' \otimes w' - v \otimes w \in A^*$ and $x' = bx$ for $b \in \mathbb{C}$. So $v' \otimes w' = v \otimes w + a^*$ for some $a^* \in A^*$. Then we have

$$\begin{aligned} y_{v' \otimes w', x'}(z) &= y_{v \otimes w + a^*, bx}(z) \\ &= \chi(\langle\langle v \otimes w + a^*, z \rangle\rangle / 2)bx \\ &= b\chi(\langle\langle a^*, z \rangle\rangle / 2)\chi(\langle\langle v \otimes w, z \rangle\rangle / 2)x \\ &= b\chi(\langle\langle a^*, z \rangle\rangle / 2)y_{v \otimes w, x}(z) \end{aligned}$$

for all $z \in -(v \otimes w) + A^*$. Hence we have $y_{v' \otimes w', x'} = cy_{v \otimes w, x}$ where $c = b\chi(\langle\langle a^*, z \rangle\rangle / 2)$. \square

Theorem 9. *With the notation as above, let k be a positive integer and let Y_k be the set of functions in Y supported on $(M^k)^* = P^{-k}A^*$. Then the following hold:*

- (i) $U_1^{2k+1}(\mathcal{A})$ and $U_2^{4k+2}(\mathcal{A})$ fix Y_k pointwise
- (ii) If f is in Y_k and $(h, 1) \in U_1^k(\mathcal{A})$ (resp. $U_2^{2k}(\mathcal{A})$), then
$$\omega_\chi(h, 1)f(v \otimes w) = \rho_A(2c(h)(v \otimes w))\chi(\langle\langle v \otimes w, c(h)(v \otimes w) \rangle\rangle)f(v \otimes w)$$

Proof. (i) Let $(g, 1) \in U_1^{2k+1}(\mathcal{A}) \cap SL_2(\mathcal{O}_F) \rtimes \{1_E\}$. Then $(g, 1) = g$ under the automorphism σ and using Proposition 7, Section 6, we will show that $\rho_A(2c(g)(v \otimes w))$ and $\chi(\langle\langle v \otimes w, c(g)(v \otimes w) \rangle\rangle)$ are trivial. We take $v \otimes w \in (P^{-k}\Gamma) \otimes (\Gamma')^*$. We will start with the latter:

$$\begin{aligned} \chi(\langle\langle v \otimes w, c(g)(v \otimes w) \rangle\rangle) &= \chi(\text{Tr}(\langle v, c(g)v \rangle \overline{\langle w, w \rangle})) \\ &= \chi(\text{Tr}(\langle v, c(g)v \rangle N_{D/F}(w))) \end{aligned}$$

Now write $g = 1 + x$, $x \in P^{2k+1}$, i.e. $x = \begin{pmatrix} x_1\varpi^{2k+1} & x_2\varpi^{2k+1} \\ x_3\varpi^{2k+1} & x_4\varpi^{2k+1} \end{pmatrix}$, then

$$\begin{aligned} c(g)v &= (1 - g)(1 + g)^{-1}v \\ &= (-x)(2 + x)^{-1}v \\ &= -2^{-1}x(1 + 2^{-1}x)^{-1}v \\ &= -2^{-1}x(1 - 2^{-1}x + (2^{-1}x)^2 - (2^{-1}x)^3 + \dots)v \\ &= -2^{-1}xv + (2^{-1}x)^2v - (2^{-1}x)^3v + \dots \\ &= \sum_{i=1}^{\infty} (-1)^i (2^{-1}x)^i v \end{aligned}$$

Hence the above trace formula will become

$$\chi(\text{Tr}(\langle v, c(g)v \rangle) N_{D/F}(w)) = \chi(\text{Tr}(\langle v, \sum_{i=1}^{\infty} (-1)^i (2^{-1}x)^i v \rangle N_{D/F}(w)))$$

Note that the term with the smallest order in above expansion is $2^{-1}xv$ and hence $\langle v, -2^{-1}xv \rangle N_{D/F}(w) = \langle a\varpi^{-k}\mathbf{u} + b\varpi^{-k}\mathbf{v}, 2^{-1}x(a\varpi^{-k}\mathbf{u} + b\varpi^{-k}\mathbf{v}) \rangle N_{D/F}(w)$ where \mathbf{u}, \mathbf{v} are the base vectors and $a, b \in \mathcal{O}_E$.

$$\begin{aligned} (1) \quad & \langle v, -2^{-1}xv \rangle = \langle a\varpi^{-k}\mathbf{u} + b\varpi^{-k}\mathbf{v}, 2^{-1}x(a\varpi^{-k}\mathbf{u} + b\varpi^{-k}\mathbf{v}) \rangle \\ (2) \quad & = \langle a\varpi^{-k}\mathbf{u} + b\varpi^{-k}\mathbf{v}, 2^{-1}x(a\varpi^{-k}\mathbf{u} + b\varpi^{-k}\mathbf{v}) \rangle \\ (3) \quad & = 2^{-1}(a\bar{a}\bar{x}_3\varpi + a\bar{x}_4\bar{b}\varpi - \bar{a}\bar{x}_1b\varpi - x_2b\bar{v}\varpi) \end{aligned}$$

Then the valuation $v((a\bar{a}\bar{x}_3\varpi + a\bar{x}_4\bar{b}\varpi - \bar{a}\bar{x}_1b\varpi - x_2b\bar{v}\varpi)N_{D/F}(w)) = 1 - 1 = 0$ and since χ has a conductor \mathcal{O}_F , this term and all the following terms with bigger valuation will vanish.

To show that the representation $\rho_A(2c(g)(v \otimes w)) = \rho_A((2c(g)v) \otimes w)$ is trivial, we consider again the valuation of involved terms: $v(c(g)v) = v(c(g)) + v(v) \geq 0$ for $g \in U_1^{2k+1}(\mathcal{A})$ and hence representation will be trivial. A similar argument shows that $U_2^{4k+2}(\mathcal{A})$ fixes Y_k pointwise. The different level is due to the ramification of D over F , i.e. $g = 1 + x, x \in P_D^{4k+2} \Rightarrow x \in P^{2k+1}$.

- (ii) Again $(g, 1) = g$ and we use Proposition 7, Section 6. $H_M = \{g \in G : (g-1)(M^k)^* \subset A^*\} = \{(g-1)P^{-k}A^* \subset A^*\}$, it follows $g \in U_1^k(\mathcal{A})$ or $U_2^{2k}(\mathcal{A})$. □

Theorem 10. *With the notation as in Theorem 9.*

- (i) *Let $(h, 1) \in U_1^{k+1}(\mathcal{A})$. If f is in Y_k , then f transforms according to ψ_{b_1} under the actions of $U_1^{k+1}(\mathcal{A})$ where $b_1 \in E^0$.*

$$b_1 = -\frac{\varpi^{-k+2}}{2}N_{D/F}(w) \begin{pmatrix} -\varpi^{-k-1}(\bar{a}b + a\bar{b}) & 2\varpi^{-k-1}a\bar{a} \\ -2\varpi^{-k-1}b\bar{b} & \varpi^{-k-1}(a\bar{b} + \bar{a}b) \end{pmatrix}$$

- (ii) Let $(h, 1) \in U_2^{2k+2}(\mathcal{A})$. If f is in Y_k , then f transforms according to ψ_{b_2} under the actions of $U_2^{2k+2}(\mathcal{A})$ where $b_2 \in D^0$.

$$b_2 = -\frac{\varpi}{2}N_{D/F}(w)(\bar{a}b - a\bar{b})$$

Proof. (i) Let $(h, 1) = h \in U_1^{k+1}(\mathcal{A})$, $h = 1 + x$, $x \in P^{k+1}$ and take $v \otimes w \in (P^{-k}\Gamma) \otimes (\Gamma')^*$. Consider the valuation of involved terms in $\rho_A((2c(h)v) \otimes w)$. As in the Theorem 9(i), the term with the smallest order in an expansion is $2^{-1}xv$ and hence its valuation $v(xv) = v(x) + v(v) = k + 1 - k = 1 \geq 0$. It follows that the ρ_A action is trivial and

$$\varphi_\chi(h, 1)f(v \otimes w) = \chi(\langle\langle v \otimes w, (c(h)v) \otimes w \rangle\rangle)f(v \otimes w)$$

Now arguing as in Theorem 9 (i), we get

$$\begin{aligned} \chi(\langle\langle v \otimes w, (c(h)v) \otimes w \rangle\rangle) &= \chi(\text{Tr}(\langle v, c(h)v \rangle N_{D/F}(w))) \\ &= \chi(\text{Tr}(\langle v, \sum_{i=1}^{\infty} (-1)^i (2^{-1}x)^i v \rangle N_{D/F}(w))) \end{aligned}$$

which is the same as

$$\chi(\text{Tr}(\langle v, -2^{-1}xv \rangle N_{D/F}(w) + \langle v, \sum_{i=2}^{\infty} (-1)^i (2^{-1}x)^i v \rangle N_{D/F}(w))).$$

The second term in above is in P_E and since χ has a conductor \mathcal{O}_F , we obtain

$$\chi(\text{Tr}(\langle v, \sum_{i=2}^{\infty} (-1)^i (2^{-1}x)^i v \rangle N_{D/F}(w))) = 1$$

and hence

$$(4) \quad \chi(\langle\langle v \otimes w, (c(h)v) \otimes w \rangle\rangle) = \chi(\text{Tr}(\langle v, -2^{-1}xv \rangle N_{D/F}(w)))$$

We will look at the computations separately. First, we want to explicitly compute $\langle v, -2^{-1}xv \rangle$.

$$xv = \begin{pmatrix} x_1\varpi^{k+1} & x_2\varpi^{k+1} \\ x_3\varpi^{k+1} & x_4\varpi^{k+1} \end{pmatrix} \begin{pmatrix} \varpi^{-k}a \\ \varpi^{-k}b \end{pmatrix}$$

which is precisely $(\varpi ax_1 + \varpi bx_2)\mathbf{u} + (\varpi ax_3 + \varpi bx_4)\mathbf{v}$.

Using this in above quadratic form will give us the explicit expression for $\langle v, -2^{-1}xv \rangle$:

$$(5) \quad \langle v, -2^{-1}xv \rangle = \langle \varpi^{-k}a\mathbf{u} + \varpi^{-k}b\mathbf{v}, -2^{-1}(\varpi ax_1 + \varpi bx_2)\mathbf{u} + (\varpi ax_3 + \varpi bx_4)\mathbf{v} \rangle$$

$$(6) \quad = -2^{-1}\varpi^{-k+1}(x_3a\bar{a} + x_4a\bar{b} - x_1\bar{a}b - x_2b\bar{b})$$

Using the result from equation (6) in equation (4), we obtain the following:

$$\begin{aligned} (7) \quad &\chi(\langle\langle v \otimes w, (c(h)v) \otimes w \rangle\rangle) = \\ &= \chi(\text{Tr}(-2^{-1}\varpi^{-k+1}(x_3a\bar{a} + x_4a\bar{b} - x_1\bar{a}b - x_2b\bar{b})N_{D/F}(w))) \\ &= \chi(-2^{-1}\varpi^{-k+1}N_{D/F}(w)(2x_3a\bar{a} + x_4a\bar{b} + x_4\bar{a}b - x_1\bar{a}b - x_1a\bar{b} - 2x_2b\bar{b})). \end{aligned}$$

This formula corresponds to the trace of the following element:

$$-2^{-1}\varpi^{-k+1}N_{D/F}(w)\begin{pmatrix}-\varpi^{-k-1}(\bar{a}b+a\bar{b}) & 2\varpi^{-k-1}a\bar{a} \\ -2\varpi^{-k-1}b\bar{b} & \varpi^{-k-1}(\bar{a}b+a\bar{b})\end{pmatrix}\begin{pmatrix}x_1\varpi^{k+1} & x_2\varpi^{k+1} \\ x_3\varpi^{k+1} & x_4\varpi^{k+1}\end{pmatrix}$$

Now putting the equation (4) and (7) together, we are able to explicitly write the formula for the traceless element b_1 involved the theta correspondence:

$$\begin{aligned} (8) \quad & \chi(-2^{-1}\varpi^{-k+1}N_{D/F}(w)(2x_3a\bar{a}+x_4a\bar{b}+x_4\bar{a}b-x_1\bar{a}b-x_1a\bar{b}-2x_2b\bar{b})) = \\ & = \chi(\text{Tr}(-2^{-1}\varpi^{-k+1}N_{D/F}(w)\begin{pmatrix}-\varpi^{-k-1}(\bar{a}b+a\bar{b}) & 2\varpi^{-k-1}a\bar{a} \\ -2\varpi^{-k-1}b\bar{b} & \varpi^{-k-1}(\bar{a}b+a\bar{b})\end{pmatrix}\begin{pmatrix}x_1\varpi^{k+1} & x_2\varpi^{k+1} \\ x_3\varpi^{k+1} & x_4\varpi^{k+1}\end{pmatrix})) \\ & = \psi(\text{Tr}(-\frac{\varpi^{-k+2}}{2}N_{D/F}(w)\begin{pmatrix}-\varpi^{-k-1}(\bar{a}b+a\bar{b}) & 2\varpi^{-k-1}a\bar{a} \\ -2\varpi^{-k-1}b\bar{b} & \varpi^{-k-1}(\bar{a}b+a\bar{b})\end{pmatrix}x)) \\ & = \psi(\text{Tr}(b_1(x))) \\ & = \psi(\text{Tr}(b_1(h-1))) \\ & = \psi_{b_1}(h) \end{aligned}$$

$$\text{where } b_1 = -\frac{\varpi^{-k+2}}{2}N_{D/F}(w)\begin{pmatrix}-\varpi^{-k-1}(\bar{a}b+a\bar{b}) & 2\varpi^{-k-1}a\bar{a} \\ -2\varpi^{-k-1}b\bar{b} & \varpi^{-k-1}(\bar{a}b+a\bar{b})\end{pmatrix}.$$

It is clear that the element b_1 is traceless.

- (ii) With the notation as above, we take $v \otimes w \in (\Gamma) \otimes (P^{-k}(\Gamma')^*)$, i.e. $v = a\mathbf{u} + b\mathbf{v}$, $w = \varpi^{-k}c + \varpi^{-k-1}d\delta$ where $a, b, c, d \in \mathcal{O}_E$. Now let $(h, 1) \in U_2^{2k+2}(\mathcal{A})$, i.e. $(h, 1) = h = 1 + x, x \in P_D^{2k+2}$.

Consider the valuation of involved terms in $\rho_A(v \otimes (2c(h)w))$. As in Theorem 9(i), the term with the smallest order in an expansion is $2^{-1}xw$ and hence its valuation $v(xw) = v(x) + v(w) = k + 1 - k - 1 = 0$. It follows that ρ_A is trivial and

$$\omega_\chi(h, 1)f(v \otimes w) = \chi(<< v \otimes w, v \otimes (c(h)w) >>)f(v \otimes w)$$

Hence we have

$$(9) \quad \chi(<< v \otimes w, v \otimes (c(h)w) >>) = \chi(\text{Tr}(< v, v > \overline{< w, c(h)w >}'))$$

$$(10) \quad = \chi(\text{Tr}((\bar{a}b - \bar{a}b)1/2\overline{\text{Tr}_{D/E}(wc(h)w)}))$$

Since χ has conductor \mathcal{O}_F , all terms vanish in $\text{Tr}_{D/E}$ will vanish but the first one. Therefore, we obtain

$$\begin{aligned} (11) \quad & \chi(\text{Tr}((\bar{a}b - \bar{a}b)1/2\overline{\text{Tr}_{D/E}(w(-2^{-1})\bar{x}w)})) = \\ & = \chi(\text{Tr}((\bar{a}b - \bar{a}b)(-1/4\overline{N_{D/F}(w)\text{Tr}_{D/E}(\bar{x})}))) \\ & = \chi(\text{Tr}((\bar{a}b - \bar{a}b)(-1/2N_{D/F}(w)(x)))) \\ & = \psi(\text{Tr}(-\frac{\varpi}{2}(\bar{a}b - \bar{a}b)N_{D/F}(w)(h-1))) \\ & = \psi_{b_2}(h) \end{aligned}$$

where $b_2 = -\frac{\varpi}{2}(\bar{a}b - \bar{a}b)N_{D/F}(w)$, clearly b_2 is traceless and an element of D . \square

Theorem 11. *With the notation as above, $N(b_2) = \det b_1$.*

Proof. In Theorem 1, $w \in (\Gamma')^*$, $w = c + \varpi^{-1}d\delta$, $c, d \in \mathcal{O}_E$ and hence $N_{D/F}(w) = N_{D/F}(c) - \varpi^{-2}N_{D/F}(d)\delta^2$. In Theorem 2, $w' \in (P^{-k}(\Gamma')^*)$, $w' = \varpi^{-k}c + \varpi^{-k-1}d\delta$ and hence $N_{D/F}(w') = \varpi^{-2k}N_{D/F}(c) - \varpi^{-2k-2}N_{D/F}(d)\delta^2 = \varpi^{-2k}N_{D/F}(w)$. So in fact, these two norms differ by a term ϖ^{-2k} which we will factor out in computations for $N(b_2)$.

$$\begin{aligned} \det b_1 &= \det\left(-\frac{\varpi^{-k+2}}{2}N_{D/F}(w) \begin{pmatrix} -\varpi^{-k-1}(\bar{a}b + a\bar{b}) & 2\varpi^{-k-1}a\bar{a} \\ -2\varpi^{-k-1}b\bar{b} & \varpi^{-k-1}(a\bar{b} + \bar{a}b) \end{pmatrix}\right) \\ &= -\frac{\varpi^{-4k+2}}{4}N_{D/F}^2(w)(\bar{a}b + a\bar{b})^2 + \varpi^{-4k+2}N_{D/F}^2(w)a\bar{a}b\bar{b} \\ &= -\frac{\varpi^{-4k+2}}{4}N_{D/F}^2(w)((\bar{a}b)^2 + 2a\bar{a}b\bar{b} + (\bar{a}b)^2 - 4a\bar{a}b\bar{b}) \\ &= -\frac{\varpi^{-4k+2}}{4}N_{D/F}^2(w)(a\bar{b} - \bar{a}b)^2 \end{aligned}$$

The right hand side is equal to

$$\begin{aligned} N(b_2) &= N\left(-\frac{\varpi}{2}(a\bar{b} - \bar{a}b)N_{D/F}(w')\right) \\ &= N\left(-\frac{\varpi}{2}(a\bar{b} - \bar{a}b)\varpi^{-2k}N_{D/F}(w)\right) \\ &= \left(-\frac{\varpi^{-2k+1}}{2}(a\bar{b} - \bar{a}b)N_{D/F}(w)\right)\overline{\left(-\frac{\varpi^{-2k+1}}{2}(a\bar{b} - \bar{a}b)N_{D/F}(w)\right)} \\ &= -\frac{\varpi^{-4k+2}}{4}N_{D/F}^2(w)(a\bar{b} - \bar{a}b)^2 \end{aligned}$$

□

Corollary 1. *Thus with the notation as above, b_1 and b_2 belong to corresponding conjugacy classes in $GL_2(F)$ and in D^\times .*

For the correspondence between conjugacy classes in $GL(n)$ and division algebra, see[15].

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